Neural Feature Learning

Lizhong Zheng

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Where Are We on the BlackBox Issue?

- Deep neural networks learn "informative functions": information/statistical measures used in loss and regulator.
- LLM and AI for Science: no model, no repetition.
- How to control, measure, or certify the internal operations?
- Aligned with many engineering applications: domain knowledge, structure, constraints, parameterized solutions.
- New mathematical tools.

A Detection Problem

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- $\mathsf{X}_1 \in$ BPSK, $\quad \mathsf{X}_2 \in$ 16-QAM;
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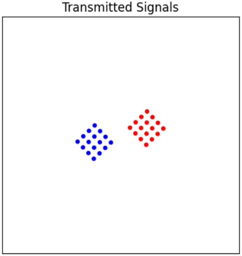

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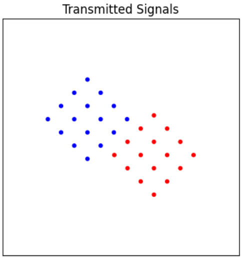

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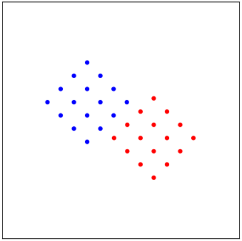
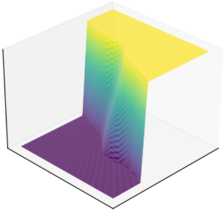
Transmitted Signals



Decision Function



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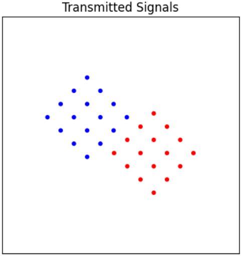

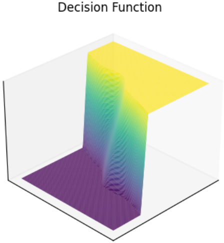

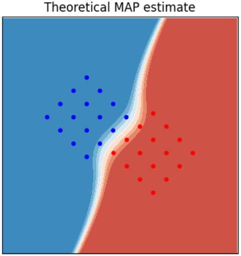

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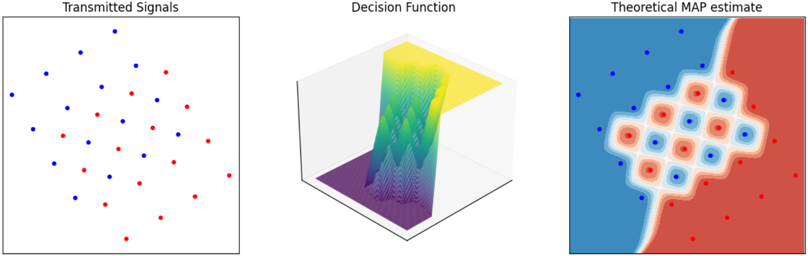

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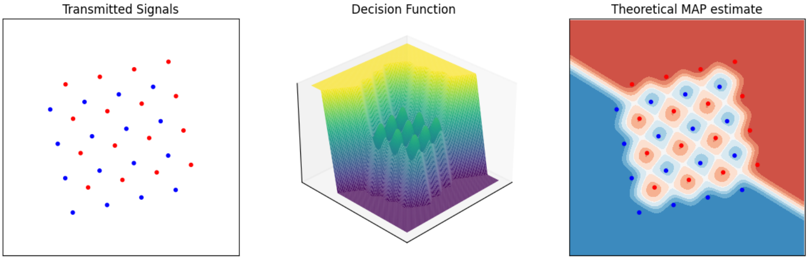

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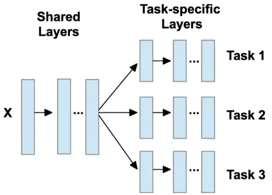

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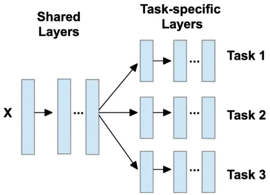
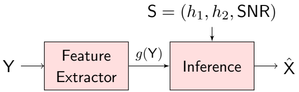


- Receiver with side-information, multi-varite dependence

$$\mathsf{Y}, \mathsf{X}, \underbrace{(h_1, h_2, \mathsf{SNR})}_{\mathsf{S}}$$



A Little Bit Tools

- Feature functions as vectors, $f : \mathcal{X} \to \mathbb{R}; \quad f \in \mathcal{F}_{\mathcal{X}}$
- Inner product

$$\langle f_1, f_2 \rangle \triangleq \mathbb{E}_{\mathsf{X} \sim R_{\mathsf{X}}}[f_1(\mathsf{X}) \cdot f_2(\mathsf{X})]$$

  - Zero-mean w.r.t. R_{X},
  - R_{X}: metric distribution, reference, ...
  - Fisher information metric

Low Rank Approximation of Dependence

- Dependence between X and Y:

$$\mathrm{i}_{\mathsf{X};\mathsf{Y}}(x,y) = \frac{P_{\mathsf{XY}}(x,y)}{P_{\mathsf{X}}(x)P_{\mathsf{Y}}(y)} - 1; \quad \mathrm{i}_{X;Y} \in \mathcal{F}_{\mathcal{X}\times\mathcal{Y}}$$

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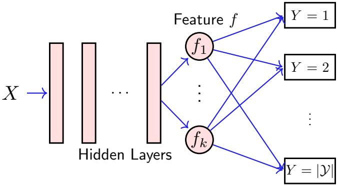
- Low-rank approximation

$$f^*, g^* = \arg \min_{f \in \mathcal{F}_{\mathcal{X}}^k, g \in \mathcal{F}_{\mathcal{Y}}^k} \left\| \mathfrak{i}_{\mathsf{X};\mathsf{Y}} - f \otimes g \right\|^2$$

$$\iff \quad P_{\mathsf{XY}}(x,y) \approx P_{\mathsf{X}}(x)P_{\mathsf{Y}}(y) \left(1 + \sum_{i=1}^{k} f_i^*(x) \cdot g_i^*(y) \right), \forall x, y$$

Feature Selection in Neural Networks



- A NN learns pairs of feature functions to form the "learned model"

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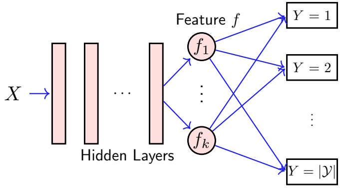

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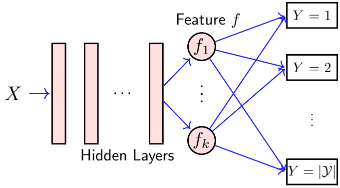

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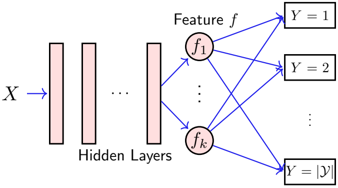
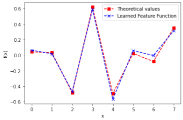
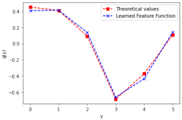




f_1



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More Directly: the H-Score Network

$$\mathscr{H}(f,g) = \|\mathsf{i}_{\mathsf{X};\mathsf{Y}}\|^2 - \|\mathsf{i}_{\mathsf{X};\mathsf{Y}} - f \otimes g\|^2$$
$$= \mathrm{cov}[f(\mathsf{X})g(\mathsf{Y})] - \frac{1}{2}\mathbb{E}[f^2(\mathsf{X})]\mathbb{E}[g^2(\mathsf{Y})]$$

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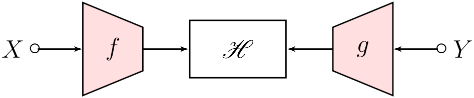
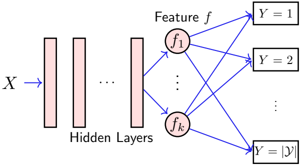
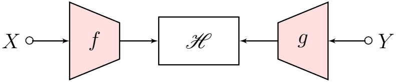
- Individual NN-modules for feature functions $f(\cdot)$ and $g(\cdot)$



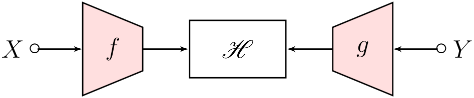
What's Good About This?





- Directly metrics of feature functions is conceptually the "right" thing to do.
  - Learn without reconstruction (weak dependence example);
  - Learn functions, not predictors.
- Control of individual feature functions.

Control of Feature Functions: Put a Constraint



constraint: $f \perp \bar{f}$

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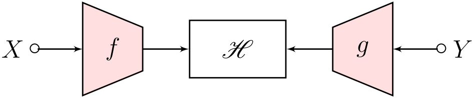

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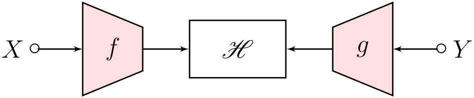

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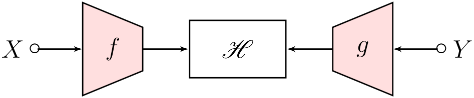

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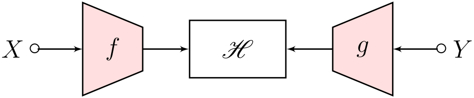

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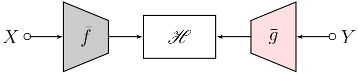
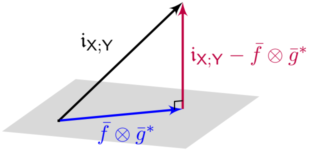
- A network that only generates f satisfying the constraint
- Generic solution without the constraint

A Projection Operation





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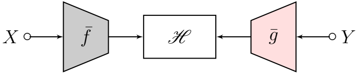

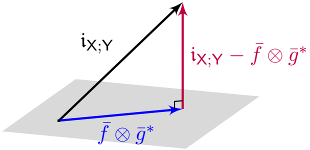

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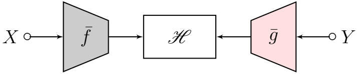

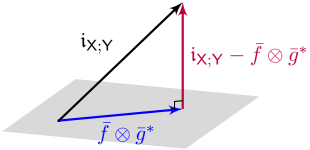

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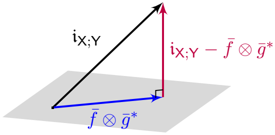
$$\min_{f,g} \left\|\left(\mathfrak{i}_{\mathsf{X;Y}} - \bar{f} \otimes \bar{g}^*\right) - f \otimes g\right\|^2$$

$$\Longleftrightarrow \min_{f,g} \left\|\mathfrak{i}_{\mathsf{X;Y}} - \left(\bar{f} \otimes \bar{g}^* + f \otimes g\right)\right\|^2$$

The Nested H-Score Network



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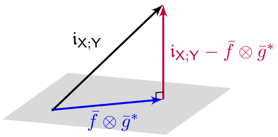

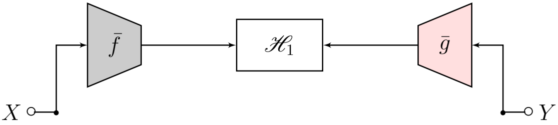

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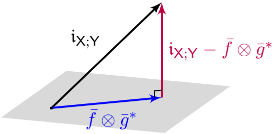

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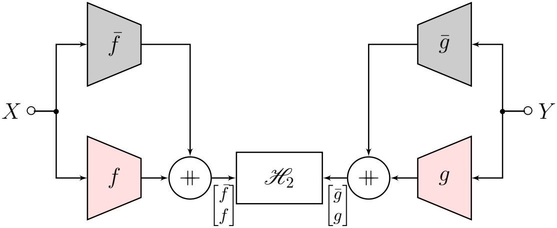

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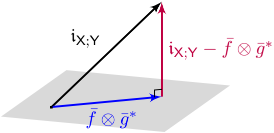

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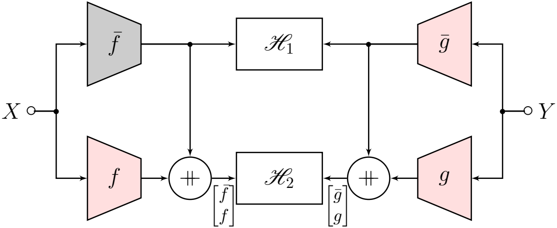
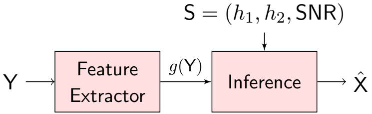


Back to The Problem

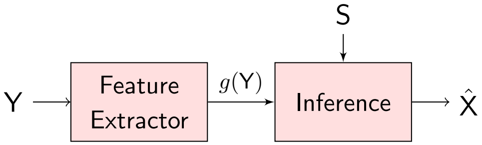
$$Y = h_1 \cdot X_1 + h_2 \cdot X_2 + W, \quad X_1 \in \{+1, -1\},\ X_2 \in 16 - \mathrm{QAM}$$

- Input is Y, output is $\hat{X}_1$.
- 3-way dependence: Y, X, S
- A decomposition



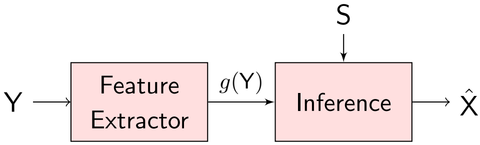
$$I(Y; (S, X)) = I(Y; S) + I(Y; X|S)$$

The Decomposition of Multi-Variate Dependence



- The three-way dependence $\mathfrak{i}_{\mathsf{Y};(\mathsf{S},\mathsf{X})}$, reference $P_{\mathsf{Y}} \cdot P_{\mathsf{S},\mathsf{X}}$,

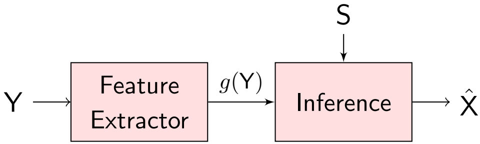
The Decomposition of Multi-Variate Dependence

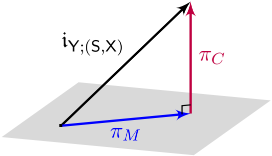


- The three-way dependence $\mathfrak{i}_{\mathsf{Y};(\mathsf{S},\mathsf{X})}$, reference $P_{\mathsf{Y}} \cdot P_{\mathsf{S},\mathsf{X}}$,
- The Markov Component

$$\pi_M = \arg \min_{\widehat{\mathfrak{i}}\,:\mathsf{Y}-\mathsf{S}-\mathsf{X}} \|\mathfrak{i}_{\mathsf{Y};(\mathsf{S},\mathsf{X})} - \widehat{\mathfrak{i}}\|^2$$

The Decomposition of Multi-Variate Dependence





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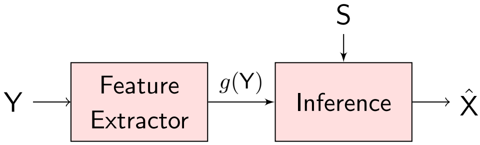
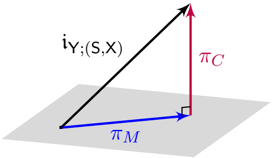
- Markov linear subspace, with conditional indep. constraints

The Decomposition of Multi-Variate Dependence





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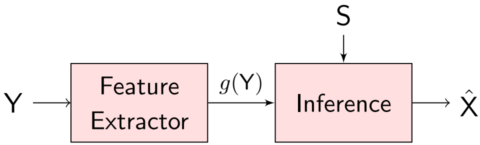
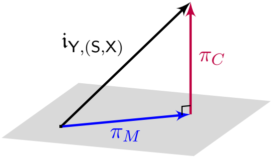
- Markov linear subspace, with conditional indep. constraints
- The Chain rule: $\underbrace{\|\mathfrak{i}_{\mathsf{Y};(\mathsf{S},\mathsf{X})}\|^2}_{I(\mathsf{Y};(\mathsf{S},\mathsf{X}))} = \underbrace{\|\pi_M\|^2}_{I(\mathsf{Y};\mathsf{S})} + \underbrace{\|\pi_C\|^2}_{I(\mathsf{Y};\mathsf{X}|\mathsf{S})}$

We Know How to Do Projections





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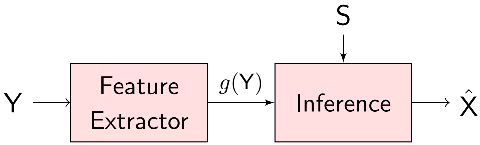

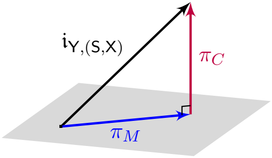

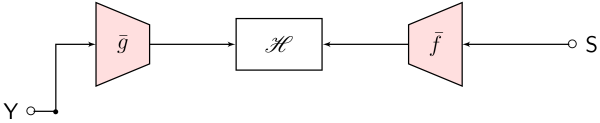

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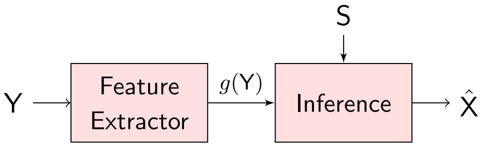

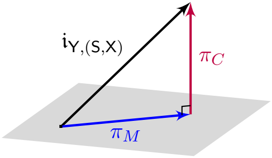

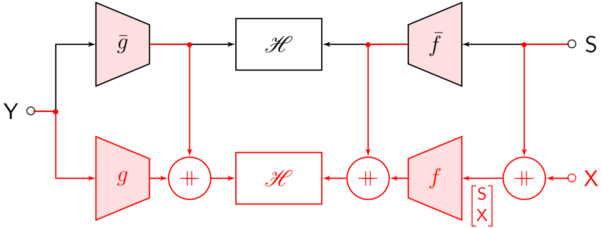

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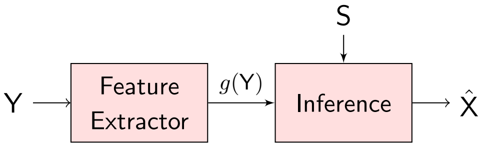
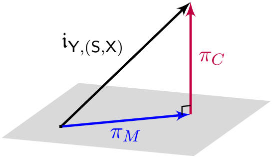
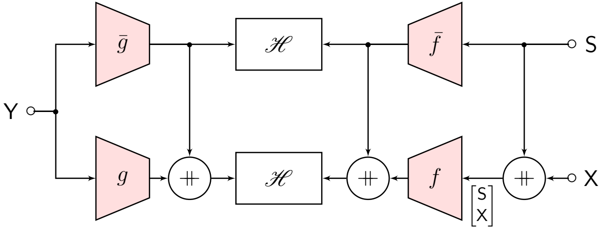
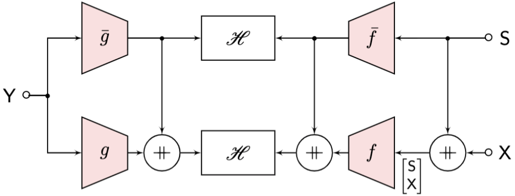




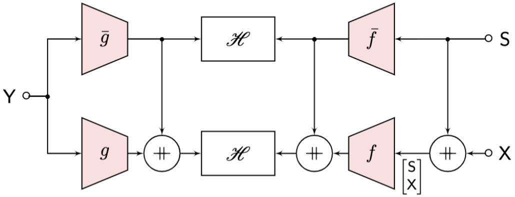


Assemble the Learning Results



- We have learned 4 feature functions, $f, g, \bar{f}, \bar{g}$

Assemble the Learning Results



- We have learned 4 feature functions, $f, g, \bar{f}, \bar{g}$
- Two good approximations:

$$P_{\mathsf{S}|\mathsf{Y}} \approx P_{\mathsf{S}} \cdot \left(1 + \bar{f}(s) \cdot \bar{g}(y)\right)$$
$$P_{\mathsf{SX}|\mathsf{Y}} \approx P_{\mathsf{SX}} \cdot \left(1 + \bar{f}(s) \cdot \bar{g}(y) + f(s, x) \cdot g(y)\right)$$

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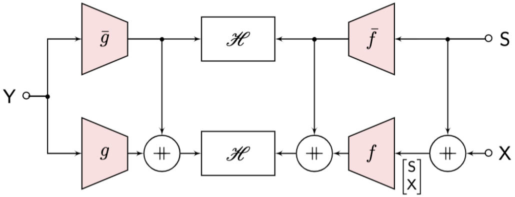

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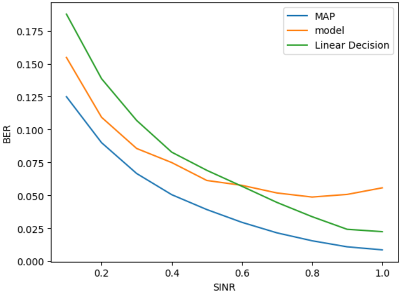
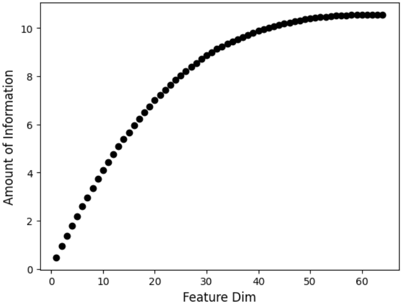
- Assemble into the predictor we need:

$$P_{\mathsf{X|S,Y}} \approx P_{\mathsf{X}} \cdot \left(1 + \frac{f(s,x) \cdot g(y)}{1 + \bar{f}(s) \cdot \bar{g}(y)}\right)$$

Performance



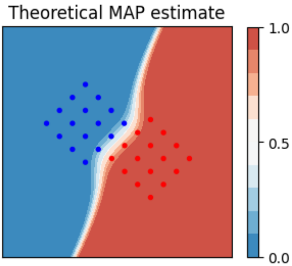
BER vs. SINR

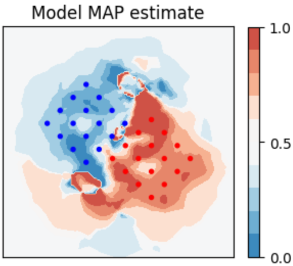


H-Score vs. Network Size

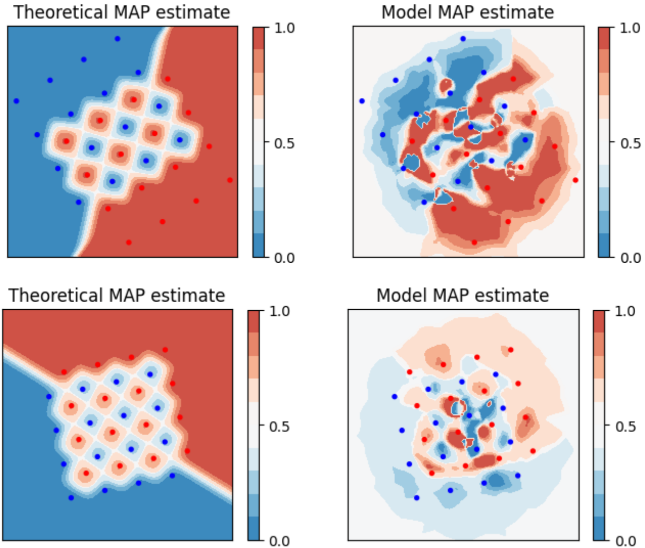
Some Special Cases

The "almost linear" case:





Harder Cases



Remarks

- Plug-and-play, no retrain, no adaptation, no few-shots, ...
- Using DNNs in engineering problems:
  - Learn feature functions, not predictors;
  - Measure quality of features, not tasks;
  - Features are high-dimensional geometric objects.

The Code

backup

$$\min_{f,g:\, f \perp \bar{f}} \left\| P_{XY} - \underbrace{P_X P_Y \cdot \left(1 + \sum_{i=1}^{k} f_i(x) \cdot g_i(y)\right)}_{\hat{P}_{XY}} \right\|^2$$